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ROCKET BOOSTER CONTROL

SECTION 5

TIME OPTIMAL CONTROL OF  
LINEAR RECURRENCE SYSTEMS

NASA Contract NASw-563

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LINEAR RECURRENCE SYSTEMS  
(NASA Contract NASw-563)

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## FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

- 1541-TR 1     Summary
- 1541-TR 2     Control of Plants Whose Representation Contains Derivatives of the Control Variable
- 1541-TR 3     Modes of Finite Response Time Control
- 1541-TR 4     A Sufficient Condition in Optimal Control
- 1541-TR 5     Time Optimal Control of Linear Recurrence Systems
- 1541-TR 6     Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
- 1541-TR 7     Penalty Functions and Bounded Phase Coordinate Control
- 1541-TR 8     Linear Programming and Bounded Phase Coordinate Control
- 1541-TR 9     Time Optimal Control with Amplitude and Rate Limited Controls
- 1541-TR 10    A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
- 1541-TR 11    A Note on System Truncation
- 1541-TR 12    State Determination for a Flexible Vehicle Without a Mode Shape Requirement
- 1541-TR 13    An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
- 1541-TR 14    Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
- 1541-TR 15    An Alternate Derivation and Interpretation of the Drift-Minimum Principle
- 1541-TR 16    A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling  $m$  components ( $1 < m \leq n$ ), of the state vector for an  $n$ -th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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TIME-OPTIMAL CONTROL OF  
LINEAR RECURRENCE SYSTEMS\*

By E. B. Lee<sup>†</sup>

ABSTRACT

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An algorithm is developed to compute the time-optimal control functions for plants represented by linear recurrence equations. Steering may be to convex sets defined by quadratic forms.

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INTRODUCTION

A computational procedure is developed to determine a control for a linear recurrence equation system so that its solution passes from an initial point to a prescribed target (set of points) in the minimum number of stages. This is an approximation to the time optimal control associated with differential equation systems. The method of solution is essentially one proposed by Ho (reference 1) for systems satisfying differential equations and modified by Eaton (reference 2). The results, in particular the notation of reference 3, will be used.

The plant model is given by the real linear recurrence equation

$$x(r+1) = A(r)x(r) + B(r)u(r) \quad (1)$$

where  $x(r)$ , an  $n$  vector, is the system state;  $u(r)$ , an  $m$  vector, is the control,  $A(r)$ ,  $B(r)$  are bounded matrices of size  $n \times n$  and  $n \times m$  respectively; and  $r = 0, 1, 2, \dots$  denotes the stage of the evolution. It is assumed that  $|u^k(r)| \leq 1$ ;  $k = 1, 2, \dots, m$ ;  $r = 0, 1, \dots$ . The initial data are denoted by  $x_0 = x(0)$  and the target set is denoted by  $G = \{x | g(x) \leq c, c \text{ is a constant}\}$ . In the work presented

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here  $g$  is assumed to be of quadratic form, that is,

$$G = \{x | x \cdot Hx \leq c, c > 0 \text{ and } H = H' > 0\}.$$

The problem to be solved requires the determination of a control sequence  $u = [u(0), u(1), \dots, u(r-1)]$  to steer the response  $x(r)$  from  $x_0$  to  $G$  in the minimum number of stages  $r$ .

### SYNTHESIS

If  $W(r)$  is such that  $W(r+1) = A(r)W(r)$ ,  $r = 0, 1, 2, \dots$ , and  $W(0) = I$ , then equation 1 can be rewritten as

$$x(r) = W(r) x_0 + W(r) \sum_{j=1}^r W^{-1}(j) B(j-1) u(j-1). \quad (2)$$

A positive definite function

$$V(x) = x \cdot Hx \quad (3)$$

and an error function

$$E(x) = x \cdot Hx - c \quad (4)$$

are introduced. The first  $r > 0$  must be determined such that  $E(x) = 0$  for some  $x(r) = x$  belonging to the set of attainability  $K(r, x_0)$ .  $K(r, x_0)$  is the collection of end points of responses  $x(r)$  which initiate at  $x_0$  for all controls  $|u^k(j)| \leq 1$ ;  $k = 1, 2, \dots, m$ ;  $j = 0, 1, \dots, r-1$ . According to reference 3,  $K(r, x_0)$  is a closed, convex, nonempty subset of  $R^n$  and therefore at each stage  $r \geq 0$  ( $r < r^* = \text{optimum } r$ ) there is a unique point  $x = x(r) \in K(r, x_0)$  which minimizes  $E(x)$ . When an  $r$ , say  $r^*$ , is found for which  $E(x) \leq 0$  for some  $x \in K(r^*, x_0)$  an optimum control sequence is known.

A computational scheme is now devised for finding the  $x \in K(r, x_0)$  which minimizes  $V(x)$  for fixed  $r > 0$ . The procedure

for increasing  $r$  is obvious.

A parameter  $t$  is introduced in equation 2 to yield

$$x(r,t) = W(r)(x_0 + \sum_{j=1}^r W^{-1}(j)B(j-1)u(j-1,t)) \quad (5)$$

From this equation

$$\dot{x}(r,t) = \sum_{j=1}^r W(r)W^{-1}(j)B(j-1) \dot{u}(j-1,t).$$

For the remaining discussion it is assumed that  $m=1$ . The restriction is easily removed.

Equation 3 is rewritten as

$$x(r,t) = W(r)x_0 + \sum_{j=1}^r h(j) u(j-1,t) \quad (7)$$

where  $h(j) = W(r)W^{-1}(j)b(j-1)$ .

$V(x(r,t))$ , and therefore  $E(x)$  will be minimized by driving each  $u(j-1,t)$  according to

$$\frac{d}{dt}u(j-1,t) = \dot{u}(j-1,t) = g^{j-1}(x(r,t)). \quad (8)$$

The functions  $g^{j-1}(x(r,t))$  will be selected in an obvious manner after the following calculation:

$$\begin{aligned} \frac{dV}{dt} &= \dot{V}(x(r,t)) = 2\dot{x}(r,t) \cdot Hx(r,t) \\ &= \sum_{j=1}^r h(j)\dot{u}(j-1,t) \cdot Hx(r,t) \\ &= 2 \sum_{j=1}^r \dot{u}(j-1,t)h(j) \cdot Hx(r,t) \end{aligned} \quad (9)$$

Then

$$\dot{u}(j-1, t) = \begin{cases} -\beta^{(j)} h(j) \cdot Hx(r, t) & \begin{cases} \text{if } |u(j-1, t)| < 1 \\ \text{if } u(j-1, t) h(j) \cdot Hx(r, t) > 0 \end{cases} \\ 0 & \text{otherwise,} \end{cases} \quad (10)$$

where each  $\beta^{(j)}$  is a positive constant. Therefore,

$$\dot{V} = -2 \sum_{j: \dot{u}(j-1, t) \neq 0} \beta^{(j)} (h(j) \cdot Hx(r, t))^2 \leq 0 \quad (11)$$

It will now be shown that  $\dot{V}$  is in fact less than zero except at  $x^*$ , the optimum point. This then establishes the requires property.

Suppose  $\dot{V}(x) = 0$  for  $x$  interior to  $K(r, x_0)$ . This can happen only if

$$u(j-1, t) = -\operatorname{sgn} \{h(j) \cdot Hx(r, t)\}$$

or

$$h(j) \cdot Hx(r, t) = 0$$

for all  $j = 0, 1, \dots, r$  ( $x = 0 \notin K(r, x_0)$ ). From reference 3 theorem 1 the response to any extremal control,  $v(j) = \operatorname{sgn}\{[W^{-1}(j)b(j-1)] \eta_0\}$  is a point on the boundary of  $K(r, x_0)$ . When  $\eta_0 = -W(r) H x(r, t)$  the above control is an extremal control and hence must lead to a response end point  $x(r, t)$  on boundary of  $K(r, x_0)$ . Therefore  $\dot{V} < 0$  if  $x(r, t)$  is interior to  $K(r, x_0)$ .

For points on the boundary of  $K(r, x_0)$ ,  $\dot{V} = 0$  only if

$$u(j-1, t) = -\operatorname{sgn}\{h(j) \cdot Hx(r, t)\}$$

or

$$h(j) \cdot Hx(r, t) = 0$$

for  $j = 1, 2, \dots, r$ . If  $x(r, t)$  is on the boundary of  $K(r, x_0)$ , then

according to reference 3 the control which gave rise to  $x(r,t)$  is necessarily an extremal control with  $\eta_r = [W^{-1}(r)]' \eta_0$  an exterior normal to  $K(r, x_0)$  at the corresponding point  $x(r,t)$  on boundary  $K(r, x_0)$ . Hence for  $\eta_0 = -W(r)H x(r,t)$  the vector

$$-[W^{-1}(r)]' W'(r) H x(r,t) = -H x(r,t)$$

must be an exterior normal to  $K(r, x_0)$  at  $x(r,t)$  on the boundary of  $K(r, x_0)$ . This can only occur if  $x(r,t) = x^*$ , because only then is  $Hx$  a vector orthogonal to the surface  $V(x) = \text{constant}$ .

Therefore  $\dot{V} < 0$  unless  $x = x^*$  and  $V(x(r,t)) \rightarrow V(x^*)$  as  $t \rightarrow \infty$ .

The only change for  $r = r^*$  is the possibility of  $x(r^*, t) \rightarrow 0$  in  $K(r^*, x_0)$ , but in this case the only point of  $K(r^*, x_0)$  at which  $\dot{V} = 0$  is still  $x^*$  with some possibility that  $x^*$  is interior to  $K(r^*, x_0)$ . For the targets under consideration  $x = 0$  is an interior point of  $G$ .

Optimum control can now be found by increasing  $r$  one step at a time and finding the point of  $K(r, x_0)$  where  $E(x)$  is a minimum. When  $E(x) \leq 0$  for  $r = r^*$  an optimum control is known.

The results apply equally well to target sets which are time varying, that is,  $G = G(r)$ .

### CONCLUSIONS

A method for making corrections to the control required for time optimal control to convex target sets has been developed. It was shown that the computation of corrections based on this scheme will converge.

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